J. Plasma Physics (1994), *vol.* 52, *part* 1, *pp.* 1–22
Copyright © 1994 Cambridge University Press

Bifurcation of BGK waves in a plasma of cold ions and electrons

By L. HANNIBAL†, E. REBHAN AND C. KIELHORN

Institut für Theoretische Physik, Heinrich-Heine-Universität Düsseldorf,
Universitätsstr. 1, D-40225 Düsseldorf, Germany

(Received 6 November 1993 and in revised form 6 May 1994)

For the simple model of cold electrons streaming against cold ions the complete set of nonlinear stationary waves is expressed in terms of elliptic functions. The conditions for their dynamical connection to a uniform neutral plasma state are taken into account, and the conditions for the neglect of the magnetic field are analysed. The range of existence of stationary waves is found to be confined to the stable regime of the two-stream instability, but covers only part of it. All nonlinear BGK waves that are found within the limits of the model can be shown to bifurcate from the two-stream instability, some of them also exhibiting secondary and further bifurcations. As an exceptional case, all bifurcations can be treated exactly. Close to the linear regime, all nonlinear modes turn out to be unstable. The corresponding instability is caused by a wave decay that transports energy from low to high wavenumbers of the Fourier modes constituting the wave. From the two-stream solutions four-stream solutions with exactly vanishing magnetic field are derived.

1. Introduction

The two-stream instability and the existence of nonlinear stationary electrostatic waves have been well known phenomena for a long time. They were studied using the simple plasma model of cold electrons streaming against cold ions: the two-stream instability by Pierce (1948) and Bohm & Gross (1949) and BGK waves by Akhiezer & Lyubarskizs (1951) and Dawson (1959). More sophisticated plasma models were also used: a waterbag model was studied (see Davidson 1960 and references therein); and Bohm & Gross (1949) employed distribution functions for treating the two-stream instability and a simple example of BGK waves, while with the same method Bernstein, Greene & Kruskal (1957) developed a full theory of the waves named after them. A recent review of nonlinear electrostatic BGK modes is that of Schamel (1986). In this paper we return to the simple cold-plasma model and closed some gaps in the existing theory. The simplicity of this model is at least partly compensated by the possibility to obtain rather far-reaching analytical results. Of course, important physical phenomena like particle trapping and Landau damping are lost this way, but, since the BGK waves obtained for non-singular distribution functions are undamped, this seems to be no great drawback. Limitations that can be attributed to the simplifications involved in the model employed in this paper will be pointed out near the end of §3.1 and in §3.5. Emphasis is put on the problem: where does the magnetic field vanish or where can it be neglected? In analogy to solutions investigated by Holloway & Dorning (1991) within the framework of a plasma model with distributed particle velocities, we also derive four-stream solutions with two cold-ion and two cold-electron streams for which the magnetic field vanishes in the whole parameter regime.

---

† Present address: Carl v. Ossietzky Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, Germany.

The paper is organized as follows. In §2 we discuss the basic equations of the problem and give a brief review of the linear theory of the basic states. In §3 we derive a compact representation of BGK waves in terms of elliptic functions‡ that is very suitable for numerical purposes. An important feature of our approach is that we take into account all conditions arising from the requirement that the nonlinear waves must be dynamically connected to the basic two-stream state. To our knowledge, these complicating conditions have never so far been taken completely into account. Although the BGK solutions can be considered to be solitary waves on a periodic interval, it will be shown in §3.4 that infinitely extended solitary waves do not exist. We shall furthermore investigate singular solutions that represent a lattice (§3.3). In §4 we study the connection between nonlinear BGK waves and the two-stream instability via bifurcations, in continuation of some previous work on this subject by one of the present authors (Rebhan 1983). Some preliminary results on the nonlinear stability of BGK waves are presented in §5. Finally, in §6 we extend our solutions to four-stream solutions with exactly vanishing magnetic field.

2. Basic equations and linear stability of the basic two-stream flow

Throughout this paper we shall employ dimensionless quantities. Since their relation to dimensional quantities will be needed for estimates, we shall briefly describe their relation to dimensional quantities (indicated by a tilde and measured in SI units). All lengths are normalized with the length $\tilde{x}_0 = 1$ m, the densities of ions and electrons with $\tilde{n}_0 = N/\tilde{x}_0^3$ where N is a typical number of electrons per cubic metre chosen such that the normalized electron density in the basic flow (see below) is equal to 1. The electrical field is normalized with the Coulomb field

$$\tilde{E}_0 = \frac{N\tilde{e}}{\tilde{\varepsilon}_0 \tilde{x}_0^2} \tag{1}$$

where $\tilde{e}$ is the electric charge of an electron and $\tilde{\varepsilon}_0$ is the vacuum dielectric constant. All velocities are normalized with the velocity defined by the equation

$$\tilde{m}_- \tilde{v}_0^2 = \tilde{e}\tilde{E}_0 \tilde{x}_0, \tag{2}$$

where $\tilde{m}_-$ is the electron mass; the time is normalized with $\tilde{t}_0 = \tilde{x}_0/\tilde{v}_0$ and magnetic fields with $\tilde{B}_0 = \tilde{E}_0/\tilde{v}_0$.

We consider perturbations of a basic flow that consists of ions with constant (normalized) density $N_+ = 1$ streaming with constant (normalized) velocity V_+ and of electrons with constant density $N_- = 1$ streaming with constant velocity V_- in the x direction. In the perturbed states the normalized densities are given by $1+n_\pm$ and the normalized velocities by $V_\pm+v_\pm$. We assume that the problem has an intrinsic periodicity in the normalized co-ordinate x and decompose the normalized electric field into its period average $\bar{E}$, which gives rise to a voltage drop over one period and a contribution that derives from a periodic potential:

---

‡ After completion of this paper, one became aware of a different representation in terms of elliptic functions by Leven & Müller (1974), described in an internal university publication.

$$E = \bar{E}-\partial_x\Phi. \tag{3}$$

With the normalization described above, the flow equations for ions and electrons and Poisson's equation for the coupling electric field are

$$\partial_t n_\pm + \partial_x[(V_\pm + v_\pm)(1+n_\pm)] = 0, \tag{4}$$

$$\partial_t v_\pm + \tfrac{1}{2}\partial_x(V_\pm + v_\pm)^2 + \left\{\begin{matrix}-\Gamma\\ 1\end{matrix}\right\}E = 0, \tag{5}$$

$$n_+ - n_- + \partial_{xx}\Phi = 0, \tag{6}$$

where $\Gamma = \tilde{m}_-/\tilde{m}_+$. We shall assume that either the magnetic field generated by the plasma current and the dielectric current is exactly zero, in which case the equation

$$d_t\bar{E}+\bar{\jmath} = 0 \tag{7}$$

(derived in the appendix) must be satisfied, or that the magnetic field must be so small that it can be neglected. The condition that must be satisfied in the latter case is

$$|d_t\bar{E}+\bar{\jmath}| \ll 0{\cdot}5\frac{10^{14}}{N}\frac{\max|n_+-n_-|}{\max|V_\pm+v_\pm|}, \tag{8}$$

where N is the typical number of electrons per cubic metre (again this is derived in the appendix). If (7) is not satisfied but (8) is then (4)–(6) provide the so-called electrostatic approximation.

In terms of the notation adopted in (4)–(6), the time-independent basic state is given by

$$n_\pm = v_\pm = \Phi = 0. \tag{9}$$

Since the equations for $n_\pm$ and $v_\pm$ are of conservation form, the mean values $\bar{n}_\pm$ and $\bar{v}_\pm$ are conserved quantities (the averages extend over one periodicity length). Since all perturbed states evolve from the basic state, we must satisfy the conditions

$$\bar{n}_\pm = 0 \quad \text{and} \quad \bar{v}_\pm = 0 \tag{10}$$

of dynamical connection.

The quantities

$$\left.\begin{aligned} w &= \frac{1}{2\Gamma}(1+n_+)(V_+ + v_+)^2 + \tfrac{1}{2}(1+n_-)(V_- + v_-)^2 + \tfrac{1}{2}E^2, \\ J &= \frac{1}{2\Gamma}(1+n_+)(V_+ + v_+)^3 + \tfrac{1}{2}(1+n_-)(V_- + v_-)^3 + (d_t\bar{E}+\bar{\jmath})\,\phi \end{aligned}\right\} \tag{11}$$

obey the equation

$$\partial_t w + \partial_x J = \bar{E}(d_t\bar{E}+\bar{\jmath}), \tag{12}$$

and can be considered as the energy density and energy flux density respectively. $\bar{w}$ is also a conserved quantity, provided that either $\bar{E} = 0$ or $d_t\bar{E}+\bar{\jmath} = 0$.

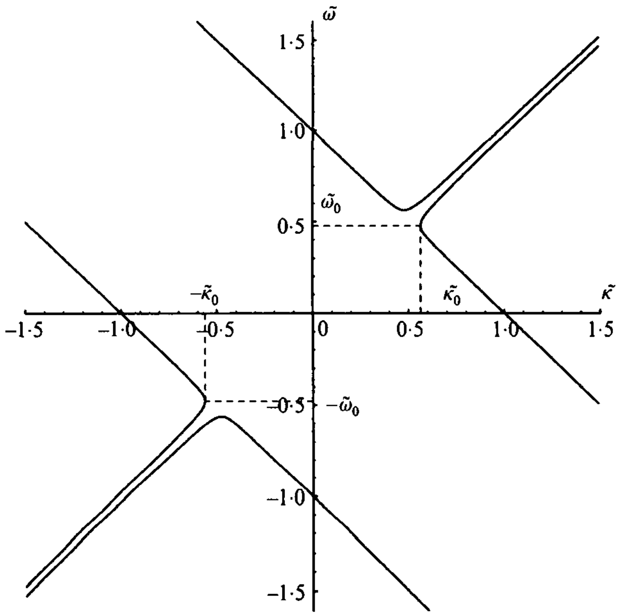

FIGURE 1. Locus of real solutions of the linear wave dispersion relation (15) in the $(\tilde{\omega}, \tilde{\kappa})$ plane.

The dispersion relation for linear waves proportional to $e^{i(\kappa x-\omega t)}$ has the well-known form

$$\frac{1}{(\omega-\kappa V_-)^2}+\frac{\Gamma}{(\omega-\kappa V_+)^2}=1. \tag{13}$$

It depends on κ only through κV_+ and κV_-, and we can shift this dependence into the definition of V_+ and V_- by setting $\kappa = 1$. With this, the periodicity length of linear waves in 2π, which will be chosen as the periodicity length of nonlinear waves as well. After the transformation

$$\tilde{\omega}=\omega-\tfrac{1}{2}(V_++V_-), \quad \tilde{\kappa}=\tfrac{1}{2}(V_+-V_-), \tag{14}$$

(13) assumes the form
$$\frac{1}{(\tilde{\omega}+\tilde{\kappa})^2}+\frac{\Gamma}{(\tilde{\omega}-\tilde{\kappa})^2}=1. \tag{15}$$

Four branches of real solutions exist, shown in figure 1 in an $(\tilde{\omega}, \tilde{\kappa})$ diagram. All four roots of (15) are real for $|\tilde{\kappa}| \geqslant |\tilde{\kappa}_0|$, when the two points on the branches where $d\tilde{\omega}/d\tilde{\kappa} = \infty$ are denoted by

$$\tilde{\kappa}=\pm\tilde{\kappa}_0, \quad \tilde{\omega}=\pm\tilde{\omega}_0. \tag{16}$$

For $-\tilde{\kappa}_0 < \tilde{\kappa} < \tilde{\kappa}_0$ only two roots $\tilde{\omega}$ are real; the other two are complex conjugates, one of them describing the two-stream instability. The latter sets on at the parameter values of the points (16). These describe marginal modes and mark bifurcation points as well.

From (15) and the condition $d\tilde{\omega}/d\tilde{\kappa} = \infty$, one easily obtains

$$\tilde{\kappa}_0=\tfrac{1}{2}(1+\Gamma^{\frac{1}{3}})^{\frac{3}{2}}, \quad \tilde{\omega}_0=\tilde{\omega}(\tilde{\kappa}_0)=\tfrac{1}{2}(1+\Gamma^{\frac{1}{3}})^{\frac{1}{2}}(1-\Gamma^{\frac{1}{3}}). \tag{17}$$

According to (14), in the original quantities we obtain marginality for

$$\omega_0=\pm\tilde{\omega}_0+\tfrac{1}{2}(V_++V_-) \tag{18}$$

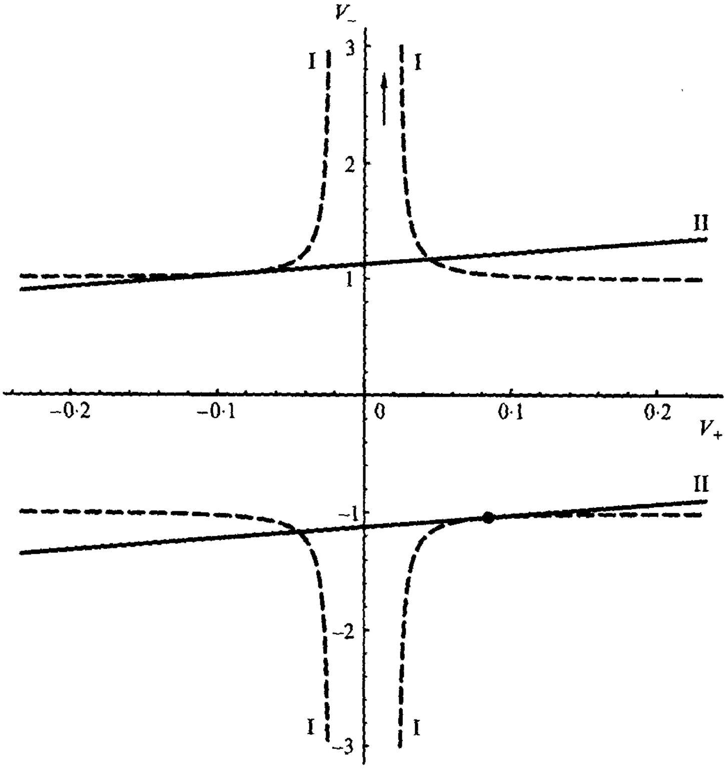

FIGURE 2. Locus of stationary (I) (---) and marginal (II) (——) modes for $k = 1$ in the (V_+, V_-) plane (● indicates V^0).

and

$$V_+ - V_- = \pm 2\tilde{\kappa}_0. \tag{19}$$

All real-ω solutions proportional to $e^{i(\kappa x - \omega t)}$ are stationary in a co-moving frame of reference; hence $\omega = 0$ can be achieved by a Galilei transformation. According to (13), stationary linear modes are obtained for

$$\frac{1}{V_-^2} + \frac{\Gamma}{V_+^2} = 1. \tag{20}$$

Stationary marginal modes are obtained by setting $\omega_0 = 0$ in (18), i.e. for

$$V_+ = \pm V_+^0 = \pm(\tilde{\kappa}_0 - \tilde{\omega}_0), \quad V_- = \mp V_-^0 = \mp(\tilde{\kappa}_0 + \tilde{\omega}_0). \tag{21}$$

In the (V_+, V_-) diagram of figure 2 they are obtained for the parameter values of the points where the curve (20) for stationary modes denoted by I and shown dashed touches the straight lines (19) for marginal modes denoted by II and shown full. (Note that the ω_0 values calculated from (18) at intersection points of the straight lines II with other branches of curve I are non-zero, i.e. they belong to a different solution branch and do not describe stationary marginal modes.)

3. Nonlinear stationary-mode solutions

3.1. *One-humped regular periodic waves*

Averaging (5) with (3) over the periodicity interval yields

$$\partial_t \bar{v}_\pm = \begin{Bmatrix} \Gamma \\ -1 \end{Bmatrix} \bar{E} \neq 0.$$

Therefore the equations for nonlinear stationary modes are obtained from (4)–(6) by dropping the time derivatives and setting $\bar{E} = 0$. With properly chosen integration constants $c_\pm$ and $W_\pm$, they can be integrated to yield

$$\left.\begin{aligned} (V_+ + v_+)(1+n_+) &= \pm(2\Gamma)^{\frac{1}{2}} c_+^2, \\ (V_- + v_-)(1+n_-) &= \pm 2^{\frac{1}{2}} c_-^2, \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} (V_+ + v_+)^2 + 2\Gamma\Phi &= 2\Gamma W_+, \\ (V_- + v_-)^2 - 2\Phi &= 2W_-, \end{aligned}\right\} \tag{23}$$

$$\partial_{xx}\Phi = n_- - n_+. \tag{24}$$

The consequences of the additional equation $j = 0$ or the condition (8) for the neglect of the magnetic field will be discussed at the end of this subsection. In (22) two possible signs of the integration constants have been introduced, admitting positive and negative signs of $V_\pm$. The signs that must be chosen in the different quadrants of the (V_+, V_-) plane can be obtained from the linear case $(v_+ \to 0, n_+ \to 0)$ by a continuity argument, and are shown in table 1, the first sign pertaining to the ions and the second to the electrons. Equations (23) can be solved to yield

$$v_+ = \pm[2\Gamma(W_+ - \Phi)]^{\frac{1}{2}} - V_+, \quad v_- = \pm[2(W_- + \Phi)]^{\frac{1}{2}} - V_-. \tag{25}$$

The signs to be chosen here can be concluded from the conditions (10), and again correspond to Table 1, where the first sign pertains once again to the ions and the second to the electrons.

Inserting (25) into (22) yields

$$n_+ = \frac{c_+^2}{(W_+ - \Phi)^{\frac{1}{2}}} - 1, \quad n_- = \frac{c_-^2}{(W_- + \Phi)^{\frac{1}{2}}} - 1 \tag{26}$$

in all four quadrants, and (24) can be rewritten as

$$\Phi''(x) + \frac{c_+^2}{(W_+ - \Phi)^{\frac{1}{2}}} - \frac{c_-^2}{(W_- + \Phi)^{\frac{1}{2}}} = 0. \tag{27}$$

Equations (25) and (26) allow the calculation of $n_\pm$ and $v_\pm$ from $\Phi(x)$ once the latter has been determined from (27).

Since $\Phi(x)$ is periodic, it must have a minimum in the periodicity interval. The x origin can be chosen such that the minimum lies at $x = 0$, whence $\Phi'(0) = 0$, and, furthermore, Φ can be gauged according to $\Phi(0) = 0$. With this, (27) can be integrated once to yield

$$[\Phi'(x)]^2 = 4c_-^2\{[W_- + \Phi(x)]^{\frac{1}{2}} - W_-^{\frac{1}{2}}\} + 4c_+^2\{[W_+ - \Phi(x)]^{\frac{1}{2}} - W_+^{\frac{1}{2}}\}. \tag{28}$$

Setting $\Phi' = 0$ then yields an equation for Φ that has two zeros, i.e. periodic

$+, +$ for $V_+ > 0, V_- > 0$	$-, +$ for $V_+ < 0, V_- > 0$
$-, -$ for $V_+ < 0, V_- < 0$	$+, -$ for $V_+ > 0, V_- < 0$

TABLE 1

solutions $\Phi(x)$ have just one minimum and one maximum value of Φ. The maximum value Φ_M is determined by the second zero of $\Phi'(x)$ and satisfies

$$\frac{W_+^{\frac{1}{2}} - (W_+ - \Phi_M)^{\frac{1}{2}}}{(W_- + \Phi_M)^{\frac{1}{2}} - W_-^{\frac{1}{2}}} = Q, \tag{29}$$

where we define Q by

$$Q = \left(\frac{c_-}{c_+}\right)^2. \tag{30}$$

From (2), we get

$$\Phi'(x) = \pm 2c_+ W(\Phi(\mathrm{x})), \tag{31}$$

where

$$W(z) = \{Q[(W_- + z)^{\frac{1}{2}} - W_-^{\frac{1}{2}}] + (W_+ - z)^{\frac{1}{2}} - W_+^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{32}$$

i.e. $\Phi'(x)$ has two branches. $\Phi(x)$ is symmetric around its extrema, and for the one-humped 2π-periodic solutions considered in this subsection the maximum is attained at $x = \pi$.

Equation (31) can be integrated once more to yield the implicit solution

$$\int_0^{\Phi(x)} \frac{dz}{W(z)} = 2c_+ x \tag{33}$$

for the first branch of $\Phi(x)$ extending from the minimum to the maximum, and a corresponding one for the second extending from the maximum to the minimum. Solutions $\tilde{\Phi}(x)$ whose first branch would extend from maximum to minimum ($\Phi'(x) < 0$) can be obtained from the latter ones as $\tilde{\Phi}(x) = \Phi(x+\pi)$ owing to the translational invariance of (27).

It is possible to express the implicit solution (33) for Φ in terms of elliptic functions. For this purpose, we first transform from $\mathrm{W}_\pm$ to new parameters

$$\alpha_\pm = \frac{W_\pm}{\Phi_M}, \tag{34}$$

in terms of which Q becomes

$$Q = \frac{\alpha_+^{\frac{1}{2}} - (\alpha_+ - 1)^{\frac{1}{2}}}{(\alpha_- + 1)^{\frac{1}{2}} - \alpha_-^{\frac{1}{2}}}. \tag{35}$$

We now go from $z = \Phi$ to a new variable ψ by setting

$$\sin\psi = \frac{\sin\frac{1}{2}u}{\sin\frac{1}{2}u_1} \tag{36}$$

where

$$u = \arcsin\left(\frac{\alpha_+ - z/\Phi_M}{\alpha_+ + \alpha_-}\right)^{\frac{1}{2}} - \alpha, \quad u_1 = \arccos\beta \tag{37}$$

with

$$\tan\alpha = \frac{1}{Q}, \quad \beta = \frac{\alpha_+^{\frac{1}{2}} + Q\alpha_-^{\frac{1}{2}}}{[(\alpha_+ + \alpha_-)(1+Q^2)]^{\frac{1}{2}}}. \tag{38}$$

The lower limit of integration $z_1 = 0$ in (33) transforms to $u_1 = \arccos\beta$ and $\psi_1 = \frac{1}{2}\pi$. Since $\sin u \leqslant \sin u_1$ for $z \geqslant 0$ according to (37), ψ is defined for all $z \leqslant \Phi_M$. From (32) and (36)–(38), we get

$$W(z) = 2^{\frac{1}{2}}[\Phi_M(\alpha_+ + \alpha_-)(1+Q^2)]^{\frac{1}{4}} \sin\tfrac{1}{2}u_1 \cos\psi \tag{39}$$

and

$$\frac{dz}{W(z)} = -\frac{2^{\frac{3}{2}}[\Phi_M(\alpha_+ + \alpha_-)]^{\frac{3}{4}}}{(1+Q^2)^{\frac{5}{4}}}\left[\frac{Q}{(1-k^2\sin^2\psi)^{\frac{1}{2}}} - 8Qk^2\sin^2\psi(1-k^2\sin^2\psi)^{\frac{1}{2}}\right.$$
$$\left. + 2(Q^2-1)(k\sin\psi - 2k^3\sin^3\psi\right]d\psi, \tag{40}$$

where k is defined by $\qquad k = \sin\frac{1}{2}u_1.$

Since $\cos u_1 = 1 - 2\sin^2\frac{1}{2}u_1 = 1 - 2k^2 = \beta$, we get, with (38),

$$k^2 = \frac{1}{2}\left\{1 - \frac{\alpha_+^{\frac{1}{2}} + Q\alpha_-^{\frac{1}{2}}}{[(\alpha_+ + \alpha_-)(1+Q^2)]^{\frac{1}{2}}}\right\}. \tag{41}$$

Using $\qquad \displaystyle\int_{\psi_1}^{\psi} d\psi' \ldots = -\int_{\psi}^{\frac{1}{2}\pi} d\psi' \ldots = -\int_0^{\frac{1}{2}\pi} d\psi' \ldots + \int_0^{\psi} d\psi' \ldots,$

$$\int_0^{\psi} k^2\sin^2\psi'(1-k^2\sin^2\psi')^{\frac{1}{2}}d\psi' = \tfrac{1}{3}(1-k^2)\int_0^{\psi}\frac{d\psi'}{(1-k^2\sin^2\psi')^{\frac{1}{2}}}$$
$$+\tfrac{1}{3}(2k^2-1)\int_0^{\psi}(1-k^2\sin^2\psi')^{\frac{1}{2}}d\psi' - \tfrac{1}{3}k^2\sin\psi\cos\psi(1-k^2\sin^2\psi)^{\frac{1}{2}},$$

and the notation

$$E(k,\psi) = \int_0^{\psi}(1-k^2\sin^2\psi')^{\frac{1}{2}}d\psi', \quad F(k,\psi) = \int_0^{\psi}\frac{d\psi'}{(1-k^2\sin^2\psi')^{\frac{1}{2}}} \tag{42}$$

for the elliptic functions, from (33) with (40) and with the abbreviations

$$E = E(k,\tfrac{1}{2}\pi), \quad F = F(k,\tfrac{1}{2}\pi),$$

we finally get the implicit representation of our solution $\Phi(x)$:

$$(2k^2-1)E(k,\psi) - (k^2-\tfrac{5}{8})F(k,\psi) - k^2\sin\psi\cos\psi(1-k^2\sin^2\psi)^{\frac{1}{2}}$$
$$+\frac{3k(Q^2-1)}{4Q}(1-2k^2+\tfrac{2}{3}k^2\cos^2\psi)\cos\psi$$
$$= (2k^2-1)E - (k^2-\tfrac{5}{8})F + \frac{3c_+(1+Q^2)^{\frac{5}{4}}}{2^{\frac{9}{2}}Q[\Phi_M(\alpha_+ + \alpha_-)]^{\frac{3}{4}}}x. \tag{43}$$

Then, according to (36)–(38), $\Phi = z$ is related to ψ by

$$\Phi = \frac{\Phi_M(\alpha_+ + \alpha_-)}{1+Q^2}\{[1-2k^2+2kQ(1-k^2)^{\frac{1}{2}}]^2$$
$$-[1-2k^2\sin^2\psi + 2kQ\sin\psi(1-k^2\sin^2\psi)^{\frac{1}{2}}]^2\}. \tag{44}$$

With the solutions thus obtained, we have to satisfy the four conditions (10) and the periodicity condition $\Phi = \Phi_M$ for $x = x_2 = \pi$. According to (36)–(38), x_2 transforms to $\psi_2 = -\frac{1}{2}\pi$, and, inserting this in (43), with $E(k, -\frac{1}{2}\pi) = -E$ and $F(k, -\frac{1}{2}\pi) = -F$, we obtain

$$(k^2-\tfrac{5}{8})F-(2k^2-1)E = \frac{3\pi c_+(1+Q^2)^{\frac{5}{4}}}{2^{\frac{9}{2}}Q[\Phi_M(\alpha_+ + \alpha_-)]^{\frac{3}{4}}}. \tag{45}$$

According to (26) with (31), the condition $\bar{n}_+ = 0$ becomes

$$c_+ = 2\pi\left[\int_0^{\phi_M} \frac{dz}{W(z)(W_+ - z)^{\frac{1}{2}}}\right]^{-1}. \tag{46}$$

With

$$\frac{dz}{W(z)(W_+ - z)^{\frac{1}{2}}} = -\frac{2^{\frac{3}{2}}[\Phi_M(\alpha_+ + \alpha_-)]^{\frac{1}{4}}}{(1+Q^2)^{\frac{3}{4}}}\,\frac{Q(1-k^2\sin^2\psi)-2k\sin\psi(1-k^2\sin^2\psi)^{\frac{1}{2}}}{(1-k^2\sin^2\psi)^{\frac{1}{2}}}\,d\psi,$$

this can be rewritten as

$$c_+ = \frac{\pi(1+Q^2)^{\frac{3}{4}}}{2^{\frac{3}{2}}Q(2E-F)[\Phi_M(\alpha_+ + \alpha_-)]^{\frac{1}{4}}}. \tag{47}$$

The condition $\bar{n}_- = 0$ is automatically satisfied if $\bar{n}_+ = 0$, because averaging (24) over one periodicity interval yields $\bar{n}_- = \bar{n}_+$. Eliminating c_+ from (45) and (47) yields

$$\Phi_M = \frac{3\pi^2(1+Q^2)^2}{64Q^2(\alpha_+ + \alpha_-)(2E-F)[(k^2-\frac{5}{8})F-(2k^2-1)E]}. \tag{48}$$

We have finally to satisfy the conditions (10), which with (25) can be rewritten as

$$V_+ = \pm\frac{(2\Gamma)^{\frac{1}{2}}}{\pi}\int_0^{\Phi_M}\frac{(W_+ - z)^{\frac{1}{2}}}{2c_+ W(z)}dz, \quad V_- = \pm\frac{2^{\frac{1}{2}}}{\pi}\int_0^{\Phi_M}\frac{(W_- + z)^{\frac{1}{2}}}{2c_+ W(z)}dz. \tag{49}$$

Since V_+ and V_- do not enter the solution $\Phi(x)$ when parametrized by $\alpha_\pm$, the last two conditions can be neglected for the moment.

Discussing the existence of solutions in the (α_+, α_-) plane now becomes very simple. From the five conditions to be satisfied by our solutions, only the first three relate Φ_M and the integration constant c_+ to the parameters α_+ and α_-. These parameters enter our solutions via the quantities Q and k^2, and the latter exist and are real if $\alpha_-^{\frac{1}{2}}$ and $(\alpha_+ - 1)^{\frac{1}{2}}$ are real, i.e. for $\alpha_- \geqslant 0$ and $\alpha_+ \geqslant 1$. The physical meaning of these conditions is the exclusion of particle trapping, because in the interior of the regime of existence they imply

$$(V_+ + v_+)^2 = 2\Gamma(W_+ - \Phi) \geqslant 2\Gamma\Phi_M(\alpha_+ - 1) > 0$$

and $(V_- + v_-)^2 = 2(W_- + \Phi) \geqslant 2\Phi_M\,\alpha_- > 0$ for all x. The boundaries of existence in the (α_+, α_-) plane are given by the two straight lines

$$\alpha_- = 0, \quad \alpha_+ = 1. \tag{50}$$

In summary, solutions can be obtained in the following way. We start by prescribing values $\alpha_- \geqslant 0$ and $\alpha_+ \geqslant 1$. For these, we calculate Q from (35), k^2 from (41) and c_+ from (47). This way, all parameters are given that are needed to determine our nonlinear solution $\Phi(x)$ from (43) and (44). The amplitude Φ_M

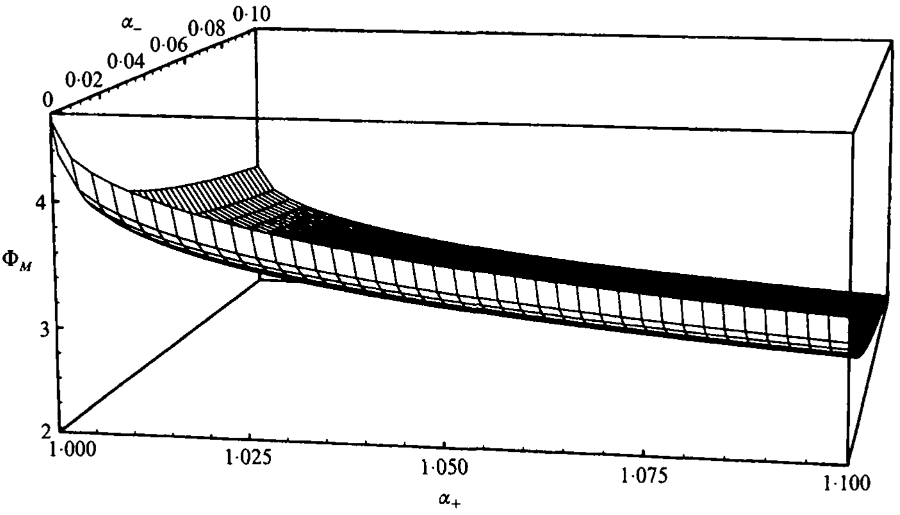

FIGURE 3. Φ_M as a function of α_- and α_+.

is shown as a function of α_- and α_+ in figure 3. The largest amplitude is obtained for $\alpha_- = 0$ and $\alpha_+ = 1$, and is given by

$$\Phi_M = \frac{3\pi^2}{2^{\frac{11}{2}}[E^2-(1+2^{-\frac{5}{2}})EF+\frac{1}{4}(1+2^{-\frac{3}{2}})F^2]} \approx 4{\cdot}638\ldots,$$

where E and F must be evaluated for $k^2 = \frac{1}{2}(1-2^{-\frac{1}{2}})$. Once $\Phi(x)$ is determined, we get $n_\pm$ from (26) after we have calculated W from (34) and c_-^2 from (30). Equations (49) can then be used to determine the original parameters $V_\pm$ that correspond to $\alpha_\pm$. Once these are determined, we finally obtain $v_\pm$ from (25).

V_+ and V_- can also be expressed in terms of elliptic functions. Using

$$(W_+-z)^{\frac{1}{2}} = \left[\frac{\Phi_M(\alpha_++\alpha_-)}{1+Q^2}\right]^{\frac{1}{2}}[1-2k^2\sin^2\psi+2kQ\sin\psi(1-k^2\sin^2\psi)^{\frac{1}{2}}],$$

$$(W_-+z)^{\frac{1}{2}} = \left[\frac{\Phi_M(\alpha_++\alpha_-)}{1+Q^2}\right]^{\frac{1}{2}}[Q(1-2k^2\sin^2\psi)-2k\sin\psi(1-k^2\sin^2\psi)^{\frac{1}{2}}],$$

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} k^2\sin^4\psi'(1-k^2\sin^2\psi')^{\frac{1}{2}}\,d\psi' = \tfrac{2}{3}[(1-k^2)F+(2k^2-1)E],$$

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} k^4\sin^4\psi'(1-k^2\sin^2\psi')^{\frac{1}{2}}\,d\psi' = \tfrac{4}{5}[(1-k^2)(2k^2-1)F+(4k^4-\tfrac{2}{3}k^2-1)E],$$

and (40), after some calculations we obtain

$$V_+ = \pm\left[\frac{2\Gamma\Phi_M(\alpha_++\alpha_-)}{1+Q^2}\right]^{\frac{1}{2}}\left\{1-\frac{2^{\frac{5}{2}}Q[\Phi_M(\alpha_++\alpha_-)]^{\frac{3}{4}}}{15\pi c_+(1+Q^2)^{\frac{5}{4}}}[(a_E+Q^2b_E)E+a_F+Q^2b_F)F]\right\},$$

$$V_- = \pm\left[\frac{2\Phi_M(\alpha_++\alpha_-)}{1+Q^2}\right]^{\frac{1}{2}}\left\{Q-\frac{2^{\frac{5}{2}}[\Phi_M(\alpha_++\alpha_-)]^{\frac{3}{4}}}{15\pi c_+(1+Q^2)^{\frac{5}{4}}}[(Q^2a_E+b_E)E+(Q^2a_F+b_F)F]\right\}, \tag{51}$$

where
$$a_E = -96k^4+56k^2-1, \quad b_E = 32k^4-32k^4+2,$$
$$a_F = 48k^4-34k^2+1, \quad b_F = -16k^4+18k^2-2.$$

Our result (51) can be used to transform the region of existence for nonlinear solutions into the original (V_+, V_-) plane. Figure 4(*a*) shows the existence regime that was obtained this way in the fourth quadrant of the (V_+, V_-) plane. According to (68) and table 1, the existence regime in the other quadrants is obtained from this by simple reflections at the V_+ and V_- axes respectively. The boundary corresponding to $\alpha_- = 0$ (or $\alpha_+ = 1$ respectively) was calculated by inserting pairs $\alpha_- = 0, \alpha_+ \geqslant 1$ (or $\alpha_- \geqslant 0, \alpha_+ = 1$ respectively) and the value Φ_M following from them in (51). An additional bounding curve is provided by the linear-mode limit $\Phi_M \to 0$. It is obviously represented by (20), and corresponds to infinity in the (α_+, α_-) plane according to (34). Basically, the curves of existence for linear solutions are broadened into band-like regions of existence for *nonlinear solutions that are confined to the stable regime of the two-stream* instability. Apparently the latter restriction is closely related to the loss of particle trapping in our plasma model.

The curves used for shading the region of existence are curves with constant $\alpha_- > 0$ or constant $\alpha_+ > 1$ respectively. It is seen that they cover some parts of it more than once. Thus, in contrast to the situation of the (α_+, α_-) plane, we have regions with multiple solutions in the (V_+, V_-) plane. Figure 4(*b*) presents a schematic view of these regions for the fourth quadrant, and indicates the multiplicity of the solutions.

In figure 5 we show what the solutions $\Phi(x)$ and the corresponding densities $1+n_\pm(x)$ look like at various points of the existence regime marked 1–6 in figure 4(*a*). We see that the densities approach a sinusoidal shape (point 1) at the left-hand boundary of the existence regime, which is due to the limit of linearity, and develop more and more pronounced peaks (points 2–6) towards the right-hand boundary. The latter is due to the appearance of singularities. According to (34), $\alpha_- = 0$ and/or $\alpha_+ = 1$ imply $W_- = 0$ and/or $\Phi_M = W_+$ respectively, and it follows from (26) and (27) that our solutions become singular then. More precisely, $n_-(x)$ and/or $n_+(x)$ become infinite (but remain integrable), whereas $\Phi(x)$ remains regular. Depending on the point of the right-hand boundary that is approached, the profile steepening can occur for electrons (point 3) and ions (point 4) separately or for both simultaneously (point 6).

We come now to the consequences of the so-far neglected equation for the magnetic field. Let us first consider the case where the latter vanishes exactly. In this case we must have

$$j = (1+n_+)(V_++v_+)-(1+n_-)(V_-+v_-) = 0,$$

which reduces to $V_+ = V_-$ for the limiting case of linear waves. With (25) and (26), the condition becomes

$$Q = \Gamma^{\frac{1}{2}}$$

in the first and third quadrants of the (V_+, V_-) plane. In accordance with the case of linear waves, the condition $Q = -\Gamma^{\frac{1}{2}}$ obtained for the second and fourth quadrants cannot be satisfied, since, according to its definition (30), Q must be positive. Figure 6 shows the regions of existence of nonlinear solutions in the first and fourth quadrants of the (V_+, V_-) plane and the curve $Q = \Gamma^{\frac{1}{2}}$ where the magnetic field vanishes exactly. Furthermore, for given values of the typical

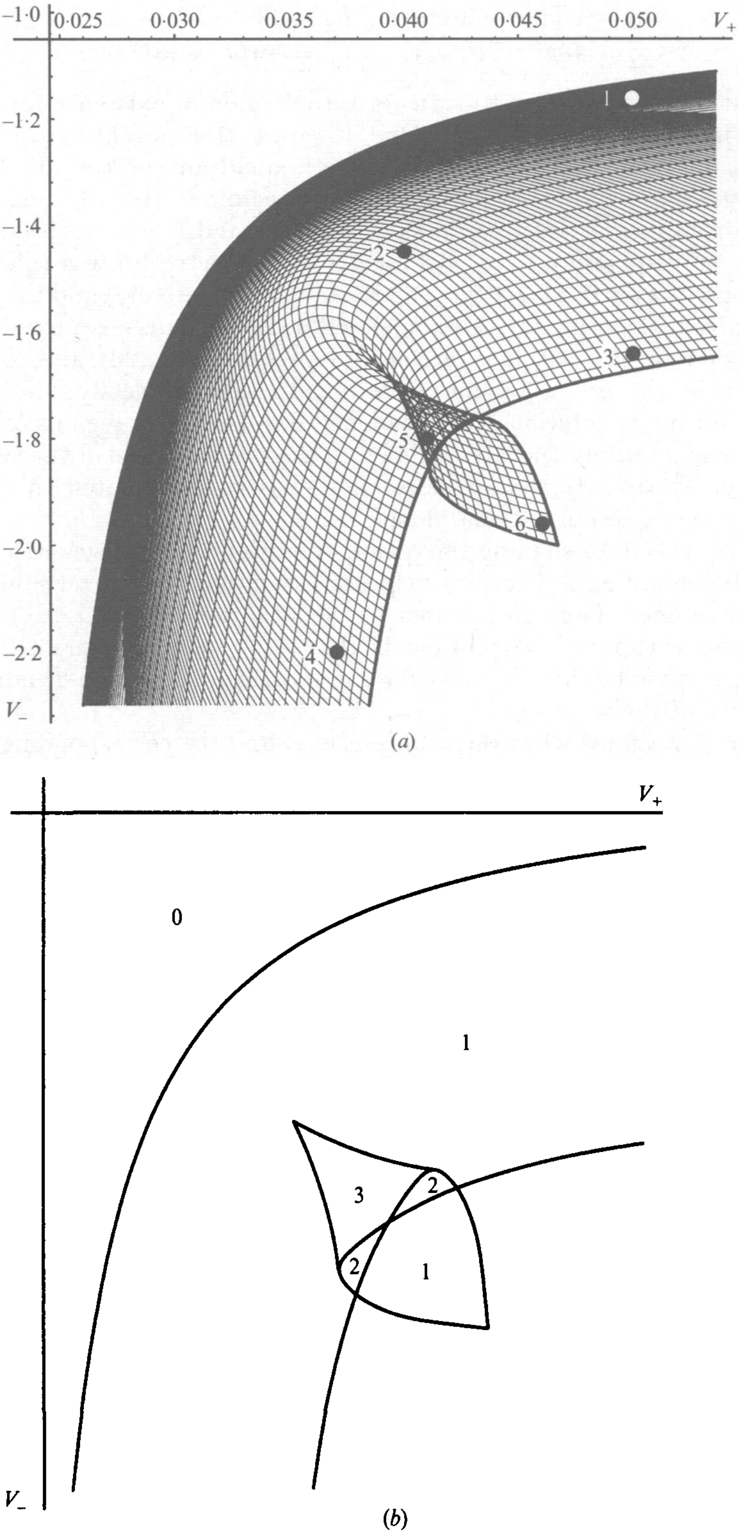

FIGURE 4. (*a*) Existence regime of nonlinear solutions in the fourth quadrant of the (V_+, V_-) plane. (Similar existence regimes can be obtained for the other three quadrants by simple reflections.) The curves used for shading the existence regime are curves with constant $\alpha_- > 0$

number N of electrons per cubic metre, it shows the regions where the magnetic field can be neglected according to (8). Corresponding domains in the second and third quadrants are obtained from these by reflection at the origin. Allowing for an error of 3×10^{-3} (i.e. $B \leqslant 3\times10^{-3}E_{typ}/max(V_\pm + v_\pm)$), with $j = 2^{\frac{1}{2}}c_+^2(\pm\sqrt{\Gamma}-Q)$ and the choice ϕ_M/π instead of $\max|n_+ - n_-|$ for the typical electric field, the boundaries of these regions are obtained from the equations

$$c_+^2\,\pi(Q\pm\Gamma^{\frac{1}{2}})\frac{\max|V_\pm + v_\pm|}{\Phi_M} = \pm\frac{10^{11}}{N}. \tag{52}$$

(On the left-hand side, for the first/second and third/fourth quadrants of the (V_+, V_-) plane the $-/+$ sign must be taken.) In the second and fourth quadrants, owing to $Q \leqslant 0$ and the $+$ sign on the left-hand side, (52) can only be satisfied for the $+$ sign on the right-hand side, and the boundary is only obtained for $N \leqslant 10^{11}/1{\cdot}77$.

In §6 we shall construct four-stream solutions from our nonlinear two-stream solutions for which the magnetic field vanishes exactly in the whole regime of existence.

3.2. *Multi-humped regular periodic waves*

Solutions with n maxima and minima in the periodicity interval $[0, 2\pi[$ can be obtained from the one-humped solutions discussed so far by a simple transformation. We take advantage of the fact that we have shifted the dependence of our solutions into the velocities $V_\pm$ by setting $\kappa = 1$. Replacing $V_\pm \rightarrow V_\pm/m$ and rescaling the functions $n_\pm, v_\pm$ and Φ properly, we obtain solutions with mean velocities $V_\pm/m$ and m maxima in the original periodicity interval of length 2π. If $n_\pm(x)$, $v_\pm(x)$ and $\Phi(x)$ with mean velocities $V_\pm$ are solutions of (4)–(6) then the rescaled functions

$$n_\pm(\kappa x), \quad \kappa v_\pm(\kappa x), \quad \kappa^2\Phi(\kappa x) \tag{53}$$

corresponding to the rescaled parameters

$$\kappa V_\pm, \quad \kappa^2 W_\pm, \quad \kappa^{\frac{1}{2}}c_\pm$$

are also solutions of (4)–(6) with periodicity length $2\pi\kappa$. Choosing $\kappa = 1/m$, we get a solution with m maxima and minima in the interval 2π. One- and multi-humped waves with equal mean velocities are only obtained when both $V_\pm$ and $V_\pm/m$ lie in the existence regime depicted in figure 4. This will not be the case in general.

3.3. *Lattice solutions*

In addition to the solutions obtained in §3.2, which are regular or limiting cases of regular ones, there exist isolated singular solutions for $V_+ = V_- = 0$ that will be considered now. Equations (22)–(24) together with the side conditions (10) can also be solved using Dirac delta distributions. It is easily seen that the function

$$\Phi(x) = \begin{cases} \pi x & (0 \leqslant x \leqslant \pi), \\ \pi(2\pi - x) & (\pi \leqslant x \leqslant 2\pi), \end{cases} \tag{54}$$

$$n_- = 2\pi\delta(x) - 1, \quad n_+ = 2\pi\delta(x-\pi) - 1, \tag{55}$$

$$v_+ = \pm(2\Gamma)^{\frac{1}{2}}[\Phi_M - \Phi(x)]^{\frac{1}{2}} - \text{const}, \quad v_- = \pm[2\Phi(x)]^{\frac{1}{2}} - \text{const}, \tag{56}$$

---

or constant $\alpha_+ > 1$. For the points marked 1–6 the profiles $\Phi(x)$, $1+n_-(x)$ and $1+n_+(x)$ are shown in figure 5. (*b*) Schematic view of the regions in (*a*) with different solution multiplicities, the latter being indicated by the numbers in each region.

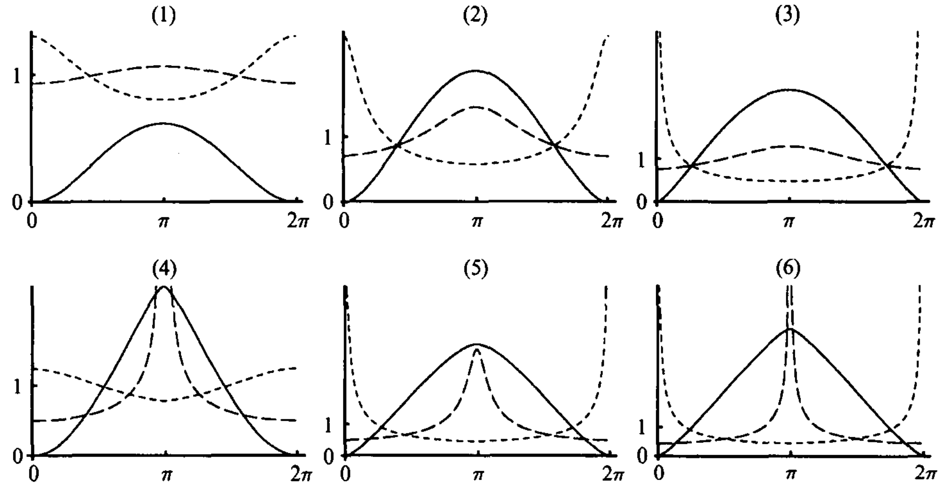

FIGURE 5. Solutions $\Phi(x)$ (——), $1+n_-(x)$ (------) and $1+n_+(x)$ (– – –) for the points numbered 1–6 in the existence regime of figure 4(*a*): (1) $V_+ = 0{\cdot}05$, $V_- = -1{\cdot}16$; (2) 0·04, −1·45; (3) 0·05, −1·64; (4) 0·037, −2·2; (5) 0·041, 7·8; (6) 0·046, −1·96.

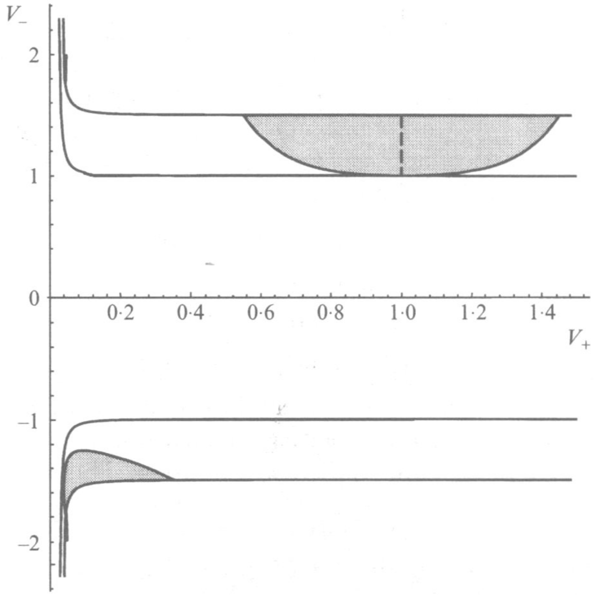

FIGURE 6. Domains of negligible magnetic field in the (V_+, V_-) plane: the magnetic field vanishes exactly on the dotted line, while in the shaded regions

$$B \leqslant 3 \times 10^{-3} E_{\mathrm{typ}} / \max\,(V_\pm + v_\pm).$$

In the first quadrant (52) yields two boundaries of the shaded region, those shown corresponding to $N = 10^{11}$ electrons m^{-3}. In the fourth quadrant only one boundary is obtained, that shown corresponding to $N = 3{\cdot}3 \times 10^{10}$ electrons m^{-3}.

is a solution in the distributional sense. From the maximum value of $\Phi(x)$, $\Phi_M = \pi^2$, it may be seen that this solution is not connected to the previous ones. It corresponds to the case where the electrons and ions are at rest in delta-distribution peaks that are regularly spaced on a lattice. Of course, each peak represents a two-dimensional plane that extends perpendicular to the lattice direction. It is easily verified that, in the same way, one can construct further lattice solutions with more and irregularly distributed peaks in one periodicity interval.

3.4. *Non-existence of infinitely extended solitary waves*

The existence of singular waves in a periodic interval would lead to regular solitary solutions of infinite extension if in (53) the spread of the interval were combined with an adequate reduction of the wave amplitude. This, however, is not the case. In this subsection we show quite generally that infinitely extended waves with a finite number of humps do not exist.

There are two possibilities for solutions of this kind:

$$\Phi \rightarrow \Phi_M \quad \text{as } |x| \rightarrow \infty \quad \text{or} \quad \Phi \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

In both cases $\Phi''(0) \rightarrow 0$ and $\Phi'(0) \rightarrow 0$ as $|x| \rightarrow \infty$ are necessary conditions, the second being satisfied automatically according to (31) and (32). From (27), (30) and (34) for $\Phi''(0) = 0$, we get

$$Q = \left(\frac{\alpha_- + \Phi/\Phi_M}{\alpha_+ - \Phi/\Phi_M}\right)^{\frac{1}{2}}. \tag{57}$$

In the first case ($\Phi \rightarrow \Phi_M$) this leads with (35) to the condition

$$\frac{\alpha_+^{\frac{1}{2}} - (\alpha_+ - 1)^{\frac{1}{2}}}{(\alpha_- + 1)^{\frac{1}{2}} - \alpha_-^{\frac{1}{2}}} = \frac{(\alpha_- + 1)^{\frac{1}{2}}}{(\alpha_+ - 1)^{\frac{1}{2}}}, \tag{58}$$

which is satisfied only on the line $\alpha_- + \alpha_+ = 0$ in the (α_-, α_+) plane. In the second case ($\Phi \rightarrow 0$) we get from (35) the condition

$$\frac{\alpha_+^{\frac{1}{2}} - (\alpha_+ - 1)^{\frac{1}{2}}}{(\alpha_- + 1)^{\frac{1}{2}} - \alpha_-^{\frac{1}{2}}} = \left(\frac{\alpha_-}{\alpha_+}\right)^{\frac{1}{2}}, \tag{59}$$

which leads again to $\alpha_- + \alpha_+ = 0$. The states satisfying this condition lie outside the range of existence ($\alpha_- \geqslant 0, \alpha_+ \geqslant 1$) for stationary waves.

The non-existence of infinitely extended solitary waves must be considered as a drawback of the plasma model employed in this paper, because such waves do exist in plasmas with distributed velocities (Schamel 1986). Since they always come along with trapped particles, the non-existence in our model is probably due to the lack of the latter.

3.5. *The direction of wave propagation*

The equation $\tilde{V}_P = (\tilde{m}_+ \tilde{V}_+ + \tilde{m}_- \tilde{V}_-)/(\tilde{m}_+ + \tilde{m}_-) = (\tilde{V}_+ + \Gamma \tilde{V}_-)/(1 + \Gamma)$ for the plasma velocity in dimensionless quantities transforms to the same equation $V_P = (V_+ + \Gamma V_-)/(1 + \Gamma)$ in dimensionless quantities. Since V_+ and V_- are the velocities of ions and electrons respectively in the rest frame of the waves, V_P is the plasma velocity in this frame. Correspondingly,

$$V_w = -\frac{V_+ + \Gamma V_-}{1 + \Gamma} \tag{60}$$

is the velocity of the waves in the rest frame of the plasma. Waves with zero velocity are obtained on the straight line $V_+ = -\Gamma V_-$, which, owing to the smallness of Γ, almost coincides with the V_+ axis of the (V_+, V_-) plane. To the right of this line all waves run to the left, and to the left of it all waves run to the right. The existence of nonlinear waves running in both directions provides the possibility for studying the collision of nonlinear waves.

4. Bifurcation of stationary nonlinear solutions

We shall now show that the nonlinear states derived in the last section bifurcate from the basic states (9) at the curve (20) that represents stationary linear modes. (For brevity, we shall restrict our analysis to the case of one-humped waves.) This way, we obtain one of the rare examples where for a rather complex system the bifurcation of states can be followed exactly over the full range of existence. Although we can use our nonlinear results directly for the treatment of the bifurcation problem, we shall nevertheless also briefly sketch the steps of a bifurcation analysis, because it allows us to follow the solutions directly in the space of the original parameters (V_+, V_-).

A genuine bifurcation analysis has to start at the critical point given by (21), and searches for nonlinear solutions on some curve that leads through this point. A possible choice for this curve would be a straight line, and the direction of it could be chosen such that it runs through any point selected in the (V_-, V_+) plane. For the moment, we shall consider only the straight line

$$V_+ = V_+^0 + \tau, \quad V_- = V_-^0 - \tau \tag{61}$$

parametrized by τ that runs under 45° from the left down to the right in the (V_+, V_-) plane. (It corresponds to the horizontal line $\tilde{\omega} = \tilde{\omega}_0$ in the diagram of figure 1.)

Since the nonlinear solutions $\Phi(x)$ can be chosen to be symmetrical with respect to $x = \pi$, a Fourier expansion of $\Phi(x)$ with cosine terms only is possible. It turns out that a bifurcation expansion of the form

$$\Phi(x) = \epsilon \cos x + \epsilon^2 \Phi^{(2)} \cos 2x + \epsilon^3 \Phi^{(3)} \cos 3x + \dots \tag{62}$$

with the small amplitude factor ϵ is successful when inserted into (22)–(24). (Note that, with this ansatz, $\Phi'(0) = 0$ automatically, so that the expansion agrees with our treatment of the full nonlinear case in the previous section.) It is seen from (25) that $n_\pm(x)$ and $v_\pm(x)$ have the same symmetry as $\Phi(x)$, whence again an expansion using only cosine terms is possible, and an ansatz of the form

$$f(x) = \epsilon f^{(1)} \cos x + \epsilon^2 f^{(2)} \cos 2x + \epsilon^3 (f^{(31)} \cos x + f^{(33)} \cos 3x) + \dots \tag{63}$$

turns out to be successful for all of them. Obviously, with this ansatz, the side conditions (10) are satisfied automatically. Finally, the integration constants contained in (22)–(25) and the driving parameter τ in (61) that describes the distance from the critical point must also be expanded in terms of ϵ, the ansatz

$$c_\pm^2 = c_{2\pm}^2 \epsilon^2 + \dots, \quad W_\pm = W_{2\pm} \epsilon^2 + \dots, \quad \tau = \tau_2 \epsilon^2 + \dots \tag{64}$$

being successful this time. Inserting all the series expansions into (22)–(24), to first order in ϵ we get the flow equations

$$n_\pm^{(1)} V_\pm^0 + v_\pm^{(1)} = 0,$$

$$v_\pm^{(1)} V_\pm^0 + \begin{Bmatrix} \Gamma \\ -1 \end{Bmatrix} = 0,$$

with solutions

$$n_+^{(1)} = \frac{\Gamma}{V_+^{0^2}}, \quad n_-^{(1)} = -\frac{1}{V_-^{0^2}}, \quad v_+^{(1)} = -\frac{\Gamma}{V_+^0}, \quad v_-^{(1)} = \frac{1}{V_-^0}, \tag{65}$$

and the electric equation $n_+^{(1)} - n_-^{(1)} + \partial_{xx}\Phi^{(1)} = 0, \qquad (66)$

which, with (62), leads to $\dfrac{1}{V_-^{0^2}} + \dfrac{\Gamma}{V_+^{0^2}} = 1 \qquad (67)$

as a solvability condition. Comparison with the condition (20) for stationary linear solutions shows that a bifurcation in the direction of the straight line (61) is possible not only from the critical point (21) but also from any point on curve I in figure 2.

To order ϵ^2, we obtain the equations

$$n_\pm^{(2)} V_\pm^0 + v_\pm^{(2)} + \tfrac{1}{2} n_\pm^{(1)} v_\pm^{(1)} = 0, \quad v_\pm^{(2)} V_\pm^0 + \tfrac{1}{4} v_\pm^{(1)^2} + \begin{Bmatrix} \Gamma \\ -1 \end{Bmatrix} \Phi^{(2)} = 0,$$

$$n_+^{(2)} - n_-^{(2)} - 4\Phi^{(2)} = 0,$$

with solutions

$$\left.\begin{aligned} n_+^{(2)} &= \frac{\Gamma}{V_+^{0^2}}\left(\frac{5}{4}\frac{\Gamma}{V_+^{0^2}} - \frac{1}{4}\right), \quad n_-^{(2)} = \frac{1}{V_-^{0^2}}\left(\frac{5}{4}\frac{1}{V_-^{0^2}} - \frac{1}{4}\right), \\ v_+^{(2)} &= \frac{1}{4}\frac{\Gamma}{V_+^0} - \frac{3}{4}\frac{\Gamma^2}{V_+^{0^3}}, \quad v_-^{(2)} = \frac{1}{4}\frac{1}{V_-^0} - \frac{3}{4}\frac{1}{V_-^{0^3}}, \end{aligned}\right\} \tag{68}$$

$$\Phi^{(2)} = \frac{1}{4}\left(\frac{\Gamma}{V_+^{0^2}} - \frac{1}{V_-^{0^2}}\right). \tag{69}$$

The direction of the bifurcation along the straight line (61) is determined by the sign of τ_2, which can only be obtained at third order in ϵ. There, the flow equations yield solutions that depend on τ_2, and inserting them in the third-order electric equation yields an equation for τ_2 with solution

$$\tau_2 = \frac{1}{16}\,\frac{(11\Gamma/V_+^{0^2} - 3)\,\Gamma^2/(V_+^0)^4 + (11/V_-^{0^2} - 3)/(V_-^0)^4}{\Gamma/V_+^{0^3} - 1/V_-^{0^3}}. \tag{70}$$

Since V_- is negative, τ_2 is positive for all points on the curve for linear stationary modes. Thus the bifurcation leads everywhere into the region where the two-stream instability is stable. According to (64), we have a pitchfork bifurcation, which means that for any wave amplitude ϵ, $-\epsilon$ is also possible, i.e. with $\Phi(\epsilon, x)$, $\Phi(-\epsilon, x)$ is also a solution. The second solutions $\Phi(-\epsilon, x) = \Phi(\epsilon, x+\pi)$ are obtained from the first ones by a trivial translation. Figure 7 shows a bifurcation diagram for the bifurcation along the straight line (61). The main-mode amplitude ϵ obtained from the bifurcation analysis is shown by the upper clear straight line, while the one extracted by Fourier analysis of the full nonlinear mode is shown by the line below this, with dots at the calculational points. The bifurcation turns around nowhere, and ends at the boundary of existence $W_- = 0$. It is thus confirmed that in the unstable regime of the two-stream instability no stationary nonlinear modes exist. Obviously each point on

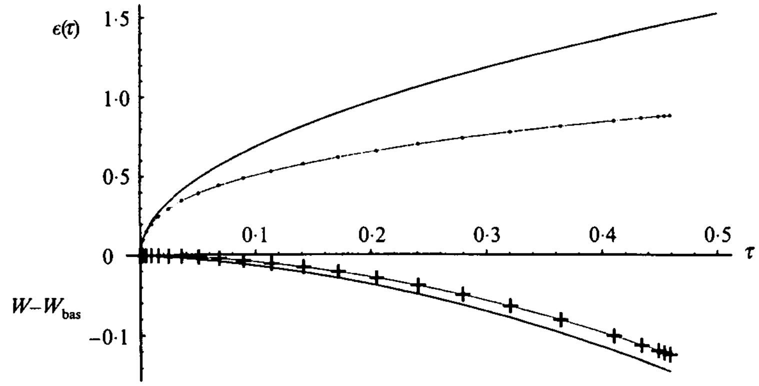

FIGURE 7. The upper two curves show the amplitude of the first cosine mode along the line (61), obtained from bifurcation analysis (——) and from the exact nonlinear solution (—·—·—). The lower two curves show the difference between the total energy of the bifurcating states and the corresponding basic state obtained from bifurcation analysis (——) and from the exact nonlinear solution —+—+—).

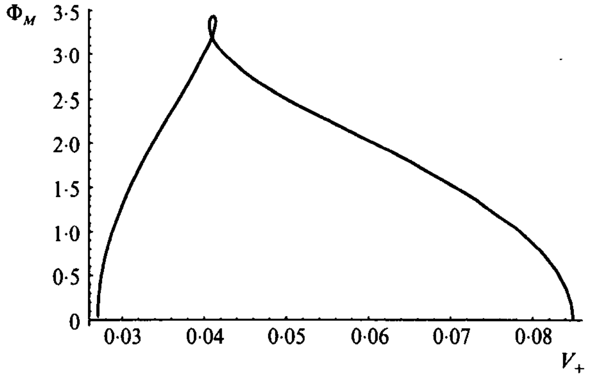

FIGURE 8. Secondary bifurcation along a straight line in the (V_+, V_-) plane crossing the region where three nonlinear solutions exist (cf. figure 4*b*).

the boundaries of the region in the (V_+, V_-) plane with two or three nonlinear solutions (see figure 4*b*) is a possible critical point for a secondary bifurcation. For example, if we give the bifurcation line (61) a different inclination such that it runs into the region with three solutions, we get a first bifurcation at its point (V_+^0, V_-^0) defined by (21) and a second one just where it crosses the boundary of this region (figure 8).

5. Stability of the nonlinear solutions

In the last section we saw that the nonlinear solutions bifurcate from the basic state into the stable region of the two-stream instability. Under certain conditions, one can conclude from this behaviour that in the vicinity of the critical point the bifurcating states are unstable and that the system tries to return from them to the corresponding basic state (same values of $V_\pm$ as the bifurcating states, but $n_\pm = 0$ and $v_\pm = 0$). This conclusion is not possible here: numerical calculation shows that the total energy $W = \int_0^{2\pi} w(x)\,dx$ contained in one periodicity interval of the bifurcating solutions is always smaller than the total energy of the corresponding basic state (see figure 7). The same result was obtained from the bifurcation analysis, yielding

$$\frac{W}{2\pi} = \frac{3}{(2\Gamma)^{\frac{1}{2}}} c_+ V_+ - \frac{3}{2^{\frac{1}{2}}} c_- V_- - 2c_+ E_+^{\frac{1}{2}} - 2c_- E_-^{\frac{1}{2}} = W_{\text{bas}} + a(V_+^0, V_-^0)\,\epsilon^4 + O(\epsilon^6), \quad (71)$$

where the energy of the basic flow is $W_{\text{bas}} = V_+^2/2\Gamma + \frac{1}{2}V_-^2$ and where $a(V_+^0, V_-^0)$ stands for a very lengthy expression that has negative values for the bifurcating states. From this, we can conclude that the system cannot try to lower its energy – provided that some friction is present – by returning to the basic state, and thus it is not possible to draw any conclusions from the bifurcational behaviour regarding stability.

So far we have carried out only a very simple stability analysis that does not allow very far-reaching conclusions. The quantities $\Phi(x)$, $n_\pm(x)$ and $v_\pm(x)$ were expanded in Fourier series

$$F(x) = \sum_{\nu=0}^{\infty} (a_{c\nu}^{(F)} \cos \nu x + a_{s\nu}^{(F)} \sin \nu x), \quad F = \Phi, n_\pm, v_\pm, \quad (72)$$

etc. and inserted into (4)–(6). The coefficients of Φ were eliminated, yielding a finite system

$$\dot{a}_{\mu\nu}^{(F)}(t) = f_{\mu\nu}^{(F)}(a_{c0}^{(\Phi)}, \ldots, a_{sN}^{(v_-)}), \quad \mu = s, c, \quad \nu = 0, \ldots, N,$$

of coupled nonlinear amplitude equations after truncation at some finite N. This system was integrated numerically with a fourth-order Runge–Kutta method. The start values of the amplitudes were chosen close to an exact nonlinear solution, i.e. at start time $t = 0$ the amplitudes $a_{s\nu}^{(\gamma)}$ were equal to zero and the $a_{c\nu}^{(\gamma)}$ were calculated by Fourier expansion of a numerically obtained nonlinear solution. Perturbations of the exact state came about by numerical uncertainty and/or a small uniform reduction or enhancement of all Fourier coefficients. Many nonlinear solutions were tested, and in the regime of small nonlinearity all turned out to be unstable. The reason for this instability was found to be a transfer of wave energy to large mode numbers. This seems to indicate either turbulent wave decay or a transition into a time-dependent state with strongly peaked waves, and will be investigated in greater detail in a forthcoming study.

6. Periodic four-stream waves with exactly vanishing magnetic field

In this section, from the two-stream solutions considered so far, we derive stationary nonlinear four-stream solutions with two ion and two electron streams for which the magnetic field vanishes exactly in the whole parameter regime. For this purpose, we assume one stream of ions/electrons with density $N_{1\pm} + n_{1\pm}$ at velocity $V_\pm + v_\pm$, and for each species a second one with density $N_{2\pm} + n_{2\pm}$ at velocity $-(V_\pm + v_\pm)$, such that the basic flow satisfies the condition

$$N_{1+} + N_{2+} = 1, \quad N_{1-} + N_{2-} = 1. \quad (73)$$

The equations corresponding to (22)–(24) are

$$(V_\pm + v_\pm)(N_{1\pm} + n_{1\pm}) = \pm(2\Gamma_\pm)^{\frac{1}{2}} c_{1\pm}^2, \tag{74}$$

$$-(V_\pm + v_\pm)(N_{2\pm} + n_{2\pm}) = \pm(2\Gamma_\pm)^{\frac{1}{2}} c_{2\pm}^2, \tag{75}$$

$$(V_+ + v_+)^2 + 2\Gamma\Phi = 2\Gamma W_+, \tag{76}$$

$$(V_- + v_-)^2 - 2\Phi = 2W_-, \tag{77}$$

$$\partial_{xx}\Phi = n_{1-} + n_{2-} - n_{1+} - n_{2+}, \tag{78}$$

where

$$\Gamma_+ = \Gamma, \quad \Gamma_- = 1. \tag{79}$$

The combination of signs must again be chosen according to table 1. The magnetic field vanishes exactly for

$$\begin{aligned} j &= (N_{1+} + n_{1+} - N_{2+} - n_{2+})(V_+ + v_+) - (N_{1-} + n_{1-} - N_{2-} - n_{2-})(V_- + v_-) \\ &= 2^{\frac{1}{2}}[\Gamma^{\frac{1}{2}}(c_{1+}^2 - c_{2+}^2) \pm (c_{1-}^2 - c_{2-}^2)] = 0, \end{aligned} \tag{80}$$

where the + sign holds in the second and fourth quadrants and the − signs in the first and third quadrants of the (V_+, V_-) plane.

From (74)–(77), we obtain

$$n_{\alpha+} = \frac{c_{\alpha+}^2}{(W_+ - \Phi)^{\frac{1}{2}}} - N_{\alpha+}, \quad n_{\alpha-} = \frac{c_{\alpha-}^2}{(W_- + \Phi)^{\frac{1}{2}}} - N_{\alpha+}, \quad \alpha = 1, 2, \tag{81}$$

and inserting this in (78) yields

$$\Phi''(x) + \frac{c_{1+}^2 + c_{2+}^2}{(W_+ - \Phi)^{\frac{1}{2}}} - \frac{c_{1-}^2 + c_{2-}^2}{(W_- + \Phi)^{\frac{1}{2}}} = 0. \tag{82}$$

For

$$c_{1+}^2 + c_{2+}^2 = c_+^2, \quad c_{1-}^2 + c_{2-}^2 = c_-^2 \tag{83}$$

this reduces to (27), and we obtain from (80) the additional condition

$$\frac{c_{1+}^2 - c_{2+}^2}{c_{1-}^2 - c_{2-}^2} = \pm\Gamma^{-\frac{1}{2}} \tag{84}$$

(with + in quadrants 1 and 3 and − in quadrants 2 and 4 of the (V_+, V_-) plane). From any two-stream solution of (27), we obtain a current-free four-stream solution by choosing c_{1+}, c_{2+}, c_{1-} and c_{2-} such that the three nonlinear equations (83) and (84) are satisfied. This is always possible, because, with the ansatz

$$c_{1\pm} = c_\pm \sin\phi_\pm, \quad c_{2\pm} = c_\pm \cos\phi_\pm,$$

(83) are satisfied and, using (30), (84) becomes

$$\frac{\cos 2\phi_+}{\cos 2\phi_-} = \pm\frac{Q}{\Gamma^{\frac{1}{2}}}.$$

Since, by proper choice of ϕ_+ and ϕ_-, the left-hand side can assume all possible values between $-\infty$ and ∞, this equation can always be satisfied.

This research was performed under the auspices of the Sonderforschungsbereich 191 of Deutsche Forschungsgemeinschaft.

Appendix. Conditions for vanishing or negligible magnetic field

If the magnetic field vanishes exactly, the Maxwell equation that relates the magnetic field to the currents becomes

$$\partial_t E + j = 0, \quad \text{where} \quad j = (1+n_+)(V_+ + v_+) - (1+n_-)(V_- + v_-), \tag{85}$$

in our dimensionless quantities. Since from (3), (4) and (6) we get

$$E = \int_0^x (n'_+ - n'_-)\,dx' - \frac{1}{2\pi}\int_0^{2\pi} dx \int_0^x dx'\,(n'_+ - n'_-) + \bar{E},$$

with the consequence $\partial_t E + j = d_t \bar{E} + \bar{j}$,

(85) is satisfied for all x if

$$d_t \bar{E} + \bar{j} = 0. \tag{86}$$

For the stationary solutions considered in this paper this condition reduces to

$$\bar{j} = 0. \tag{87}$$

When (87) is not satisfied, the current j will generate a magnetic field $\mathbf{B}$. In that case we consider a plasma cylinder with radius r whose axis extends in the x direction and we assume cylindrical symmetry of the magnetic field. Instead of (86), we then get

$$B = \tfrac{1}{2}\alpha(d_t \bar{E} + \bar{j})\, r, \quad \text{with} \quad \alpha = \left(\frac{\tilde{v}_0}{\tilde{c}}\right)^2, \tag{88}$$

where B is the poloidal magnetic field at radius r and $\tilde{c}$ is the velocity of light. The magnetic field would have to be taken into account in the momentum equations (5). It is strictly negligible if

$$|(V_\pm + v_\pm)B| \ll |E|, \tag{89}$$

for all x, and it can be neglected on the average if the inequality holds for a typical E_{typ}, which can be obtained from (6) as

$$E_{\text{typ}} \approx \max|n_+ - n_-|.$$

With (88), $r \approx 1$ and $\alpha \approx 3{\cdot}5N \times 10^{-14}$

according to (1) and (2), where N is the typical number of charges per cubic metre, the condition (89) for the neglect of the magnetic field finally becomes

$$|d_t \bar{E} + \bar{j}| \ll 0{\cdot}5\,\frac{10^{14}\max|n_+ - n_-|}{N\max|V_\pm + v_\pm|}. \tag{90}$$

REFERENCES

Akheizer, A. I. & Lyubarskizs, G. Ya. 1951 *Dokl. Akad. Nauk SSSR* **80**, 193.

Bernstein, I. B., Greene, J. M. & Kruskal, M. D. 1957 *Phys. Rev.* **108**, 546.

Bohm, D. & Gross, E. P. 1949 *Phys. Rev.* **75**, 1851, 1864.

Davidson, R. C. 1960 *Methods in Nonlinear Plasma Theory*. Academic.

Dawson, J. M. 1959 *Phys. Rev.* **113**, 383.

Holloway, J. P. & Dorning, J. J. 1991 *Phys. Rev.* A **44**, 3856.

JACKSON, E. A. 1960 *Phys. Fluids* **3**, 786.

LEVEN, R. & MÜLLER, W. F. 1974 *Wiss. Zeitschr. Ernst-Moritz-Arndt-Universität.*

PIERCE, J. R. 1948 *J. Appl. Phys.* **19**, 231.

REBHAN, E. 1983 *Proceedings of 16th Conference on Phenomena in Ionized Gases, Düsseldorf* (ed. W. Bötticher, H. Wenk & E. Schulz-Gulde), p. 688. Organising Committee ICPIG XVI.

SCHAMEL, H. 1986 *Phys. Rep.* **140**, 161.